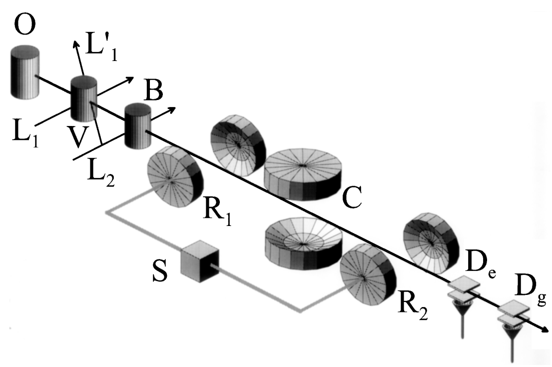
Physics 566: Quantum Optics I
Problem Set 7
Due Friday, November 15, 2013 (5:00 PM)

Problem 1: Cavity QED (20 points)
In this problem we will explore some of Prof. Serge Haroche's seminal quantum optics experiments in cavity QED for which he was awarded the Nobel Prize in 2012. These demonstrate the quantum nature of the electromagnetic field – here microwave photons, as far from high energy photons as you can get!

Consider the following schematic:



Rubidium atoms, effusing from the oven *O*, are prepared at a well-defined time and selected with a well defined speed in *V*. An atom can then be prepared in box *B* in one of two circular Rydberg states, $|g\rangle$ or $|e\rangle$, with principle quantum numbers *n*=50, 51, respectively, $\omega_{eg} = 51$ GHz. The atom passes through a high-Q superconducting cavity *C* such that one of the cavity modes, with frequency ω_c, is tuned near to the atomic transition $|e\rangle \leftrightarrow |g\rangle$. After passing through the cavity, an atom can measured in detectors that determine if it is in state $|g\rangle$ or $|e\rangle$.

In addition, the quantum cavity *C* is sandwiched inside a *Ramsey Interferometer*. The Ramsey separated two zones, R_1 and R_2, correspond to classical microwave pulses that can apply $\pi/2$ – pulses on the $|e\rangle \leftrightarrow |g\rangle$ transition. In contrast to the standard atomic clock that we have studied, in which there is free evolution between the zones, here we have a quantum super-high-Q quantum cavity in the middle.

(a) Consider the case of a stream of atoms initially prepared in *B* in the excited state $|e\rangle$ and the cavity *C* in the vacuum. The cavity is tuned to resonance, $\omega_c = \omega_{eg}$ and the vacuum Rabi frequency is $2g_0/(2\pi) = 50$ kHz. The atoms are sent through the cavity and interact for a time *t* and then measured. The atomic beam flux is chosen very low, so that the separation time between atoms crossing the cavity, *T*, is much longer than the lifetime of a photon in *C*, so each atom sees a fresh vacuum. The Ramsey zones, R_1 and R_2, are note used in this experiment.

What is the probability of detecting the atom in the state $|g\rangle$ after it passes through *C* as a function of the interaction time. Sketch this and comment on your result.

(b) This apparatus can be used to detect the presence or absence of a single photon by looking at the correlation between two atoms that pass through the cavity. Consider the same operating conditions as part (a). The velocity is now chosen so that the interaction time is $2g_0 t = \pi$. A second atom is sent through the cavity in state $|g\rangle$ for the same interaction time.

Show the conditional probability of measuring the second atom in $|e\rangle$ conditioned on measuring the first in $|g\rangle$ is e^{-T/τ_c} where T is the time separation of the two atoms, and τ_c is the cavity decay time. Comment on this result.

(c) Now let's employ the Ramsey cavities. One can use this to demonstrate the *transfer of quantum coherence* between two atoms, mediated by the quantum mode of the cavity. The operating conditions are again the same as above. With the quantum cavity C initially in the vacuum, the first Ramsey zone R_1 applies a $\pi/2$-pulse around x to the atom and prepares it in the superposition $\left(|e\rangle + i|g\rangle\right)/\sqrt{2}$. This atom passes through the quantum cavity C for an interaction time $2g_0 t = \pi$ and then measured to be in the state $|e\rangle$ or $|g\rangle$. After a time T, a second atom, initially in the state $|g\rangle$, is sent through the quantum cavity C for an interaction time $2g_0 t = \pi$. The second Ramsey zone R_2, with field phased-shifted by ϕ relative to the pulse in R_1, acts to apply $\pi/2$-pulse to this second atom. We read out the state of the two atoms.

Show the conditional probability of measuring the second atom in $|e\rangle$ conditioned on measuring the first in $|g\rangle$ is $(1+e^{-T/(2\tau_c)}\cos\phi)/2$. Give a Bloch sphere description of the transfer of coherence between the two atoms.

(d) A Ramsey interferometer can be used to measure the light-shift on a atom, as we have studied. Here we want to measure the *light shift of the quantized field* and show how this can be used to measure the absence or presence of a photon without destroying it (a so-called quantum nondemolition (QND) measurement). Suppose now that the cavity is slight *detuned* from resonance $\Delta = \omega_c - \omega_{eg} >> g_0$. The atoms are prepared in R_1 in $\left(|e\rangle + i|g\rangle\right)/\sqrt{2}$, and passed through the cavity with exactly n photons inside. The speed is sufficiently slow so that the initial "bare states" $|e,n\rangle$ and $|g,n\rangle$ adiabatically follow the "dressed states" of the coupled atom+cavity. The joint state after the interaction is $|n\rangle\left(e^{-i\delta_{e,n}}|e\rangle + ie^{-i\delta_{g,n}}|g\rangle\right)/\sqrt{2}$, where $\delta_{e,n}$ and $\delta_{g,n}$ are the phase shifts imparted to the states due to the light-shift (dressed) interaction. Note, the cavity still has exactly n photons – after the atom emerges, it neither absorbed or emitted a photons, but the quantized field caused a rotation of the atomic state in its Bloch sphere.

Find $\delta_{e,n}$ and $\delta_{g,n}$ and design the experiment to measure the photon number n.

Problem 4: The relative role of vacuum fluctuations and radiation (20 points) – Extra Credit
As discussed in lecture, spontaneous emission and level shifts can be ascribed to the perturbing effect of vacuum fluctuations and/or radiation reaction. These are actually two sides of the same coin – how we apportion the phenomenon to vacuum fluctuations vs. radiation reaction depends on how we analyze the problem. In this problem we will fill in a few details.

(a) Starting with the fundamental Hamiltonian for a two-level atom coupled to the vacuum in the dipole and rotating wave approximation, find the Heisenberg equations of motion show that they can be written as

$$\frac{d}{dt}\hat{a}_{\mathbf{k}\mu} = -i\omega_k \hat{a}_{\mathbf{k}\mu} - i g^*_{\mathbf{k}\mu}\hat{\sigma}_-$$
$$\frac{d}{dt}\hat{\sigma}_+ = i\omega_{eg}\hat{\sigma}_+ - i\sum_{\mathbf{k}\mu} g^*_{\mathbf{k}\mu}\left[s\,\hat{a}^\dagger_{\mathbf{k}\mu}\hat{\sigma}_z + (1-s)\hat{\sigma}_z\,\hat{a}^\dagger_{\mathbf{k}\mu}\right]$$
$$\frac{d}{dt}\hat{\sigma}_z = -2i\sum_{\mathbf{k}\mu}\left(g_{\mathbf{k}\mu}\left[s\hat{\sigma}_+\hat{a}_{\mathbf{k}\mu} + (1-s)\hat{a}_{\mathbf{k}\mu}\hat{\sigma}_+\right]\right) + h.c.$$

Here the s is a parameter that we will choose in the range, $0 \le s \le 1$ (don't confuse this with the saturation parameter). When s=1, all photon annihilation operators are to the right and all photon creation operators are to the left – the equations are said to be in *"normal order."* When s=0, the opposite is true and the equation are said to be in *"anti-normal order."* When s=1/2, the equations are said to be in "*symmetric order*." The choice of s determines the way in which we apportion level shifts and spontaneous decay to vacuum fluctuations vs. radiation reaction.

(b) Show that to first order in the coupling constant (Born approximation),

$$\hat{a}_{\mathbf{k}\mu}(t) = \hat{a}^{free}_{\mathbf{k}\mu}(t) + \delta\hat{a}_{\mathbf{k}\mu}(t), \quad \hat{\sigma}_+(t) = \hat{\sigma}^{free}_+(t) + \delta\hat{\sigma}_+(t), \quad \hat{\sigma}_z(t) = \hat{\sigma}^{free}_z(t) + \delta\hat{\sigma}_z(t)$$

where

$$\hat{a}^{free}_{\mathbf{k}\mu}(t) = \hat{a}_{\mathbf{k}\mu}(0)e^{-i\omega_k t}, \quad \delta\hat{a}_{\mathbf{k}\mu}(t) = -i g^*_{\mathbf{k}\mu}\int_0^t e^{-i\omega_k(t-t')}\hat{\sigma}^{free}_-(t') = -i g^*_{\mathbf{k}\mu}\,\zeta(\omega_k - \omega_{eg})\hat{\sigma}^{free}_-(t)$$
$$\hat{\sigma}^{free}_+(t) = \hat{\sigma}_+(0)e^{i\omega_{eg}t}$$
$$\delta\hat{\sigma}_+(t) = -i\int_0^t dt' e^{i\omega_{eg}(t-t')}\sum_{\mathbf{k}\mu} g^*_{\mathbf{k}\mu}\hat{a}^{\dagger free}_{\mathbf{k}\mu}(t')\hat{\sigma}^{free}_z(t') = -i\sum_{\mathbf{k}\mu} g^*_{\mathbf{k}\mu}\,\zeta(\omega_k - \omega_{eg})\hat{a}^{\dagger free}_{\mathbf{k}\mu}(t)\hat{\sigma}^{free}_z(t)$$
$$\hat{\sigma}^{free}_z(t) = \hat{\sigma}_z(0)$$

$$\delta\hat{\sigma}_z(t) = -2i\int_0^t dt' \sum_{\mathbf{k}\mu} g_{\mathbf{k}\mu}\hat{\sigma}^{free}_+(t')\hat{a}^{free}_{\mathbf{k}\mu}(t') + h.c. = -2i\sum_{\mathbf{k}\mu} g_{\mathbf{k}\mu}\zeta^*(\omega_k - \omega_{eg})\hat{\sigma}^{free}_+(t)\hat{a}^{free}_{\mathbf{k}\mu}(t) + h.c.$$

With $\zeta(\omega_{eg} - \omega_k) = \int_0^t e^{i(\omega_{eg}-\omega_k)(t-t')}\,dt' \approx \pi\delta(\omega_{eg} - \omega_k) + iP\left[\frac{1}{\omega_{eg} - \omega_k}\right]$ (the Markoff approx.).

Notes: $\hat{a}^{free}_{\mathbf{k}\mu}(t)$ is the "vacuum field" and $\delta\hat{a}_{\mathbf{k}\mu}(t)$ the "source field" leading to radiation reaction.

Because the free field commute at equal times we need not worry about operator ordering here.

(c) Take the Heisenberg state of the joint atom field system to be $|\Psi\rangle_{AF} = |\psi\rangle_A \otimes |0\rangle_F$, i.e. an arbitrary state of the atom and the field in the vacuum. Using the perturbation expansion, show that the expected values of the observables evolve according to

$$\frac{d}{dt}\langle\hat{\sigma}_+\rangle = i\omega_{eg}\langle\hat{\sigma}_+\rangle - i\sum_{\mathbf{k}\mu} g^*_{\mathbf{k}\mu}\left[s\left\langle\delta\hat{a}^\dagger_{\mathbf{k}\mu}\hat{\sigma}_z^{free}\right\rangle + (1-s)\left(\left\langle\delta\hat{\sigma}_z\,\hat{a}^{free\dagger}_{\mathbf{k}\mu}\right\rangle + \left\langle\hat{\sigma}_z^{free}\delta\hat{a}^\dagger_{\mathbf{k}\mu}\right\rangle\right)\right]$$

$$\frac{d}{dt}\langle\hat{\sigma}_z\rangle = -2i\sum_{\mathbf{k}\mu}\left(g_{\mathbf{k}\mu}\left[s\left\langle\hat{\sigma}_+^{free}\delta\hat{a}_{\mathbf{k}\mu}\right\rangle + (1-s)\left(\left\langle\hat{a}^{free}_{\mathbf{k}\mu}\delta\hat{\sigma}_+\right\rangle + \left\langle\delta\hat{a}_{\mathbf{k}\mu}\hat{\sigma}_+^{free}\right\rangle\right)\right]\right) + h.c.$$

Note the relative contributions of the vacuum field and the source field depending on the operator order we had initially chosen.

(d) Put this all together to show

$$\left(\frac{d}{dt} - i\omega_{eg}\right)\langle\hat{\sigma}_+\rangle = s\underbrace{\left(-\frac{\Gamma}{2} - i\delta\right)}_{radiation-reaction}\langle\hat{\sigma}_+\rangle + (1-s)\left(\underbrace{-2\left(\frac{\Gamma}{2} + i\delta\right)}_{vacuum\,contribution} + \underbrace{\left(\frac{\Gamma}{2} + i\delta\right)}_{radiation-reaction}\right)\langle\hat{\sigma}_+\rangle = -\left(\frac{\Gamma}{2} + i\delta\right)\langle\hat{\sigma}_+\rangle$$

$$\frac{d}{dt}\langle\hat{\sigma}_z\rangle = s\underbrace{\left(-\Gamma\langle\hat{\sigma}_z\rangle - \Gamma\right)}_{radiation-reaction} + (1-s)\left(\underbrace{-2\Gamma\langle\hat{\sigma}_z\rangle}_{vacuum\,contribution} + \underbrace{\left(\Gamma\langle\hat{\sigma}_z\rangle - \Gamma\right)}_{radiation-reaction}\right) = -\Gamma\langle\hat{\sigma}_z\rangle - \Gamma$$

$$\text{where}\;\; \Gamma = 2\pi\sum_{\mathbf{k}\mu}\left|g_{\mathbf{k}\mu}\right|^2\delta(\omega_k - \omega_{eg}), \quad \delta = \sum_{\mathbf{k}\mu} P\left[\frac{\left|g_{\mathbf{k}\mu}\right|^2}{\omega_{eg} - \omega_k}\right]$$

<u>We learn for this the following lessons:</u>
(i) The decay of populations and coherences can be calculated in the Heisenberg picture.
(ii) The way we apportion the relative contributions to levels shifts arising from vacuum fluctuations and radiation reaction depends on operator ordering, but the total result is independent of operator ordering, as expected.
(iii) In normal ordering we attribute the whole of the level shift and decay rate to radiation reaction.
(iv) In antinormal ordering both vacuum fluctuations and radiation reaction contribute to both level shifts and decay rates.
(v) In symmetric ordering, the entire level shift is due to vacuum fluctuations, but decay has contributions from vacuum fluctuation and radiation reaction.
(vi) In no ordering is the decay attributable solely to vacuum fluctuations.